Difermion condensates in vacuum in 2-4D four-fermion interaction models*

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In any four fermion (denoted by q) interaction models, the couplings of $(qq)^2$-form can always coexist with the ones of $(\bar{q}q)^2$-form via the Fierz transformations. Hence, even in vacuum, there could be interplay between the condensates $\langle\bar{q}q\rangle$ and $\langle qq\rangle$. Theoretical analysis of this problem is generally made by relativistic effective potentials in the mean field approximation in 2D, 3D and 4D models with two flavor and N_c color massless fermions.

It is found that in ground states of these models, interplay between the two condensates mainly depend on the ratio G_S/H_S for 2D and 4D case or G_S/H_P for 3D case, where G_S, H_S and H_P are respectively the coupling constants in a scalar $(\bar{q}q)$, a scalar (qq) and a pseudoscalar (qq) channel.

In ground states of all the models, only pure $\langle\bar{q}q\rangle$ condensates could exist if G_S/H_S or G_S/H_P is bigger than the critical value $2/N_c$, the ratio of the color numbers of the fermions entering into the condensates $\langle qq\rangle$ and $\langle\bar{q}q\rangle$. Below it, differences of the models will manifest themselves.

In the 4D Nambu-Jona-Lasinio (NJL) model, as G_S/H_S decreases to the region below $2/N_c$, one will first have a coexistence phase of the two condensates then a pure $\langle qq\rangle$ condensate phase. Similar results come from a renormalized effective potential in the 2D Gross-Neveu model, except that the pure $\langle qq\rangle$ condensates could exist only if $G_S/H_S = 0$. In a 3D Gross-Neveu model, when $G_S/H_P < 2/N_c$, the phase transition similar to the 4D case can arise only if $N_c > 4$, and for smaller N_c, only a pure $\langle qq\rangle$ condensate phase exists but no coexistence phase of the two condensates happens. The $G_S - H_S$ (or $G_S - H_P$) phase diagrams in these models are given.

The results deepen our understanding of dynamical phase structure of four-fermion interaction models in vacuum. In addition, in view of absence of difermion condensates in vacuum of QCD, they will also imply a real restriction to any given two-flavor QCD-analogous NJL model, i.e. in the model, the derived smallest ratio G_S/H_S via the Fierz transformations in the Hartree approximation must be bigger than 2/3.

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I. MAIN RESULTS

We have researched interplay between the fermion(q)-antifermion ($\bar{q}$) condensates $\langle\bar{q}q\rangle$ and the difermion condensates $\langle qq\rangle$ in vacuum in 2D, 3D and 4D four-fermion interaction models with two flavor and N_c color massless fermions. It is found that the ground states of the systems could be in different phases shown in the following $G_S - H_S$ and $G_S - H_P$ phase diagrams [Fig.(a)–Fig.(d)], where

G_S — coupling constant of scalar $(\bar{q}q)^2$ channel

H_S — coupling constant of scalar color $\dfrac{N_c(N_c-1)}{2}$ −plet $(qq)^2$ channel (4D, 2D case)

H_P — coupling constant of pseudoscalar color $\dfrac{N_c(N_c-1)}{2}$ −plet $(qq)^2$ channel (3D case)

Λ — Euclidean Momentum cutoff of loop integrals (4D,3D case)

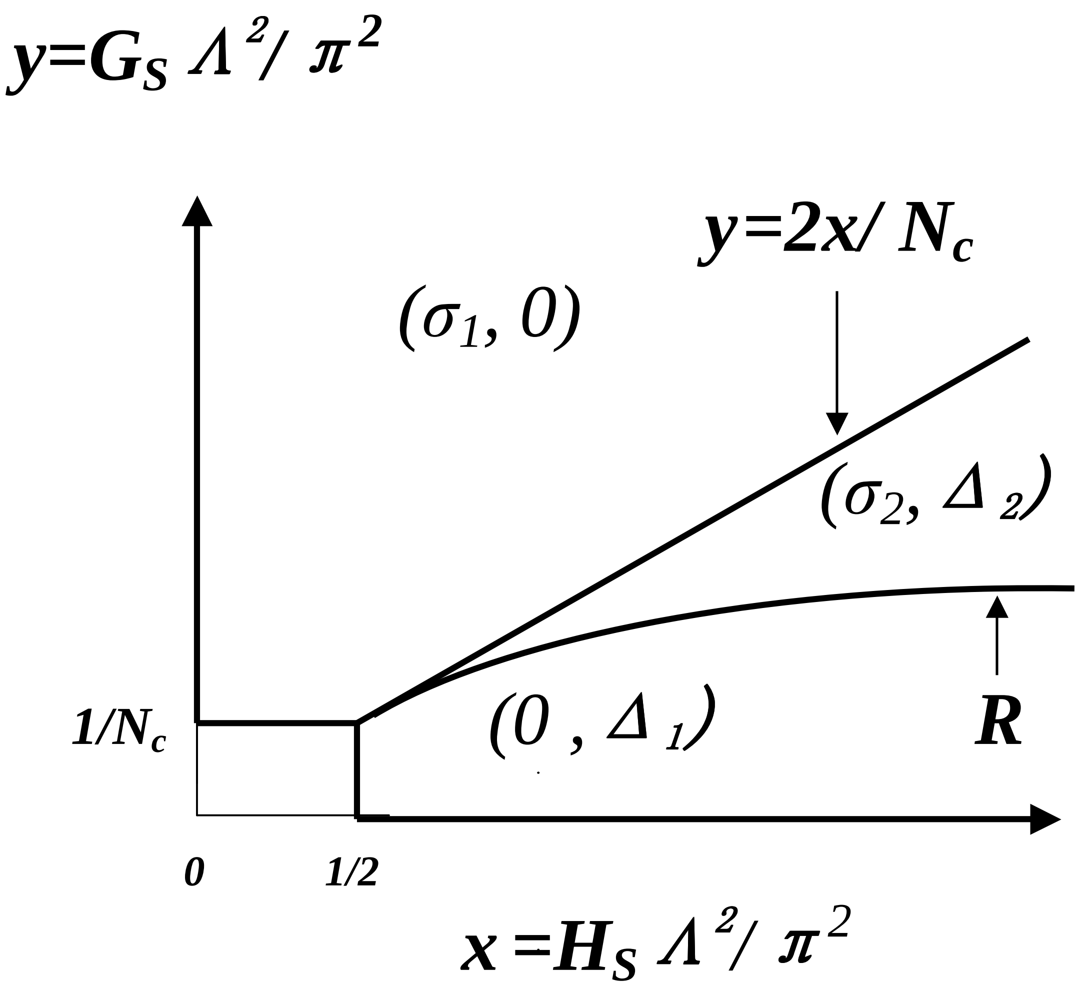
$(\sigma_1, 0)$ — pure $\langle\bar{q}q\rangle$ phase

$(0, \Delta_1)$ — pure $\langle qq\rangle$ phase

(σ_2, Δ_2) — mixed phase with both $\langle\bar{q}q\rangle$ and $\langle qq\rangle$

Fig.(a)-Fig.(d) (pages 3-6)

4D NJL Model



$$R: \quad y=x/[1+(N_c-2)x]$$

Fig. (a)

2D GN Model

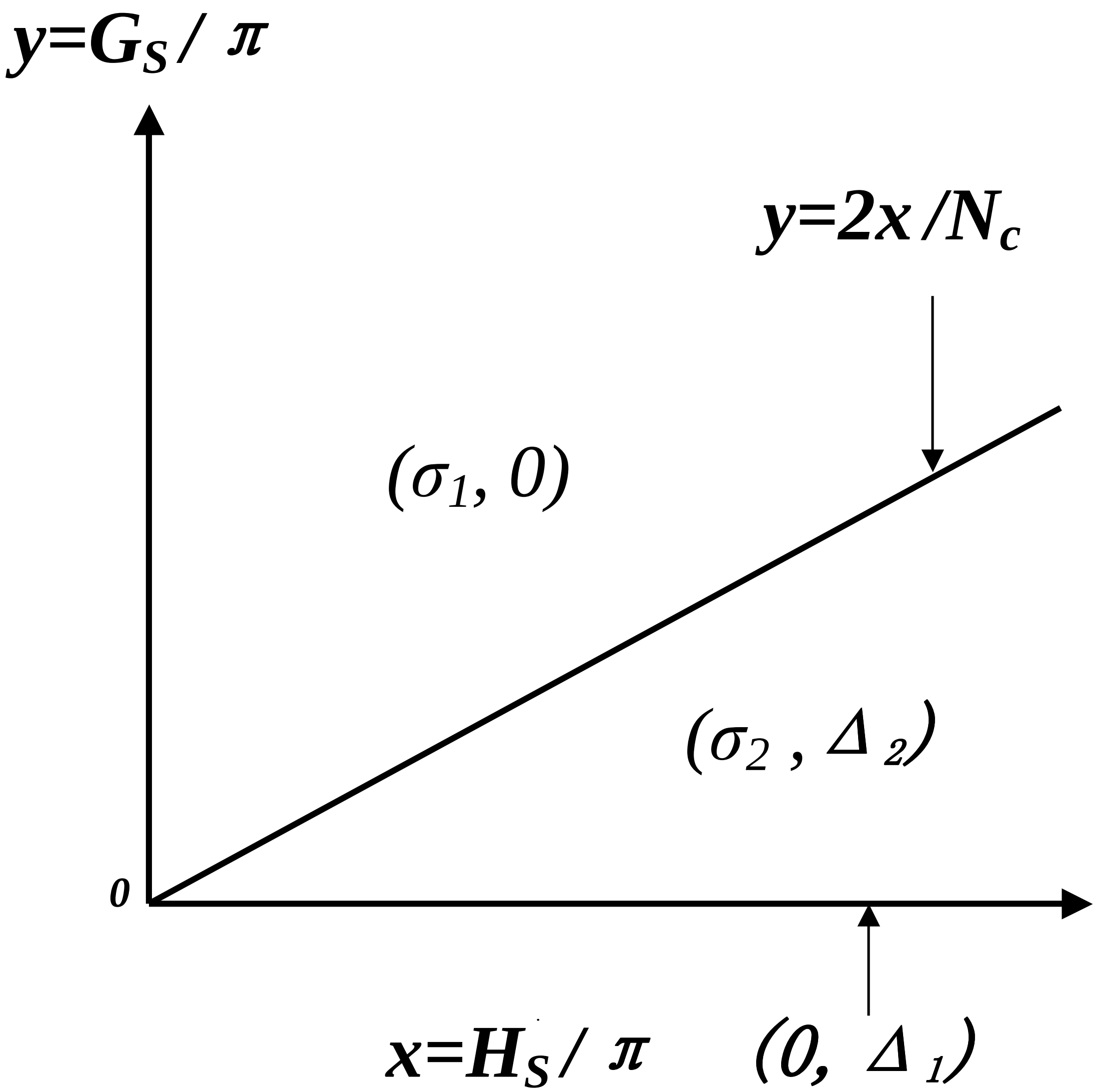


Fig. (b)

3D GN Model, $N_c \leq 4$

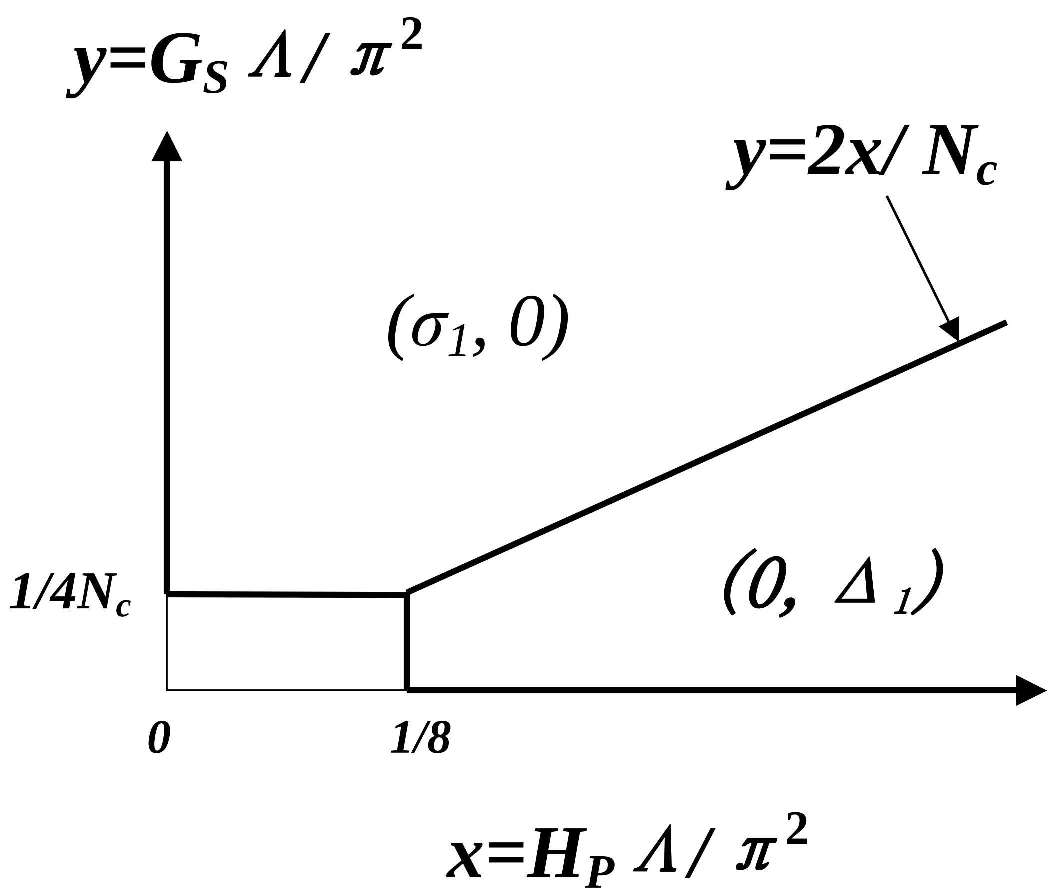


Fig. (c)

3D GN Model, $N_c \geq 5$

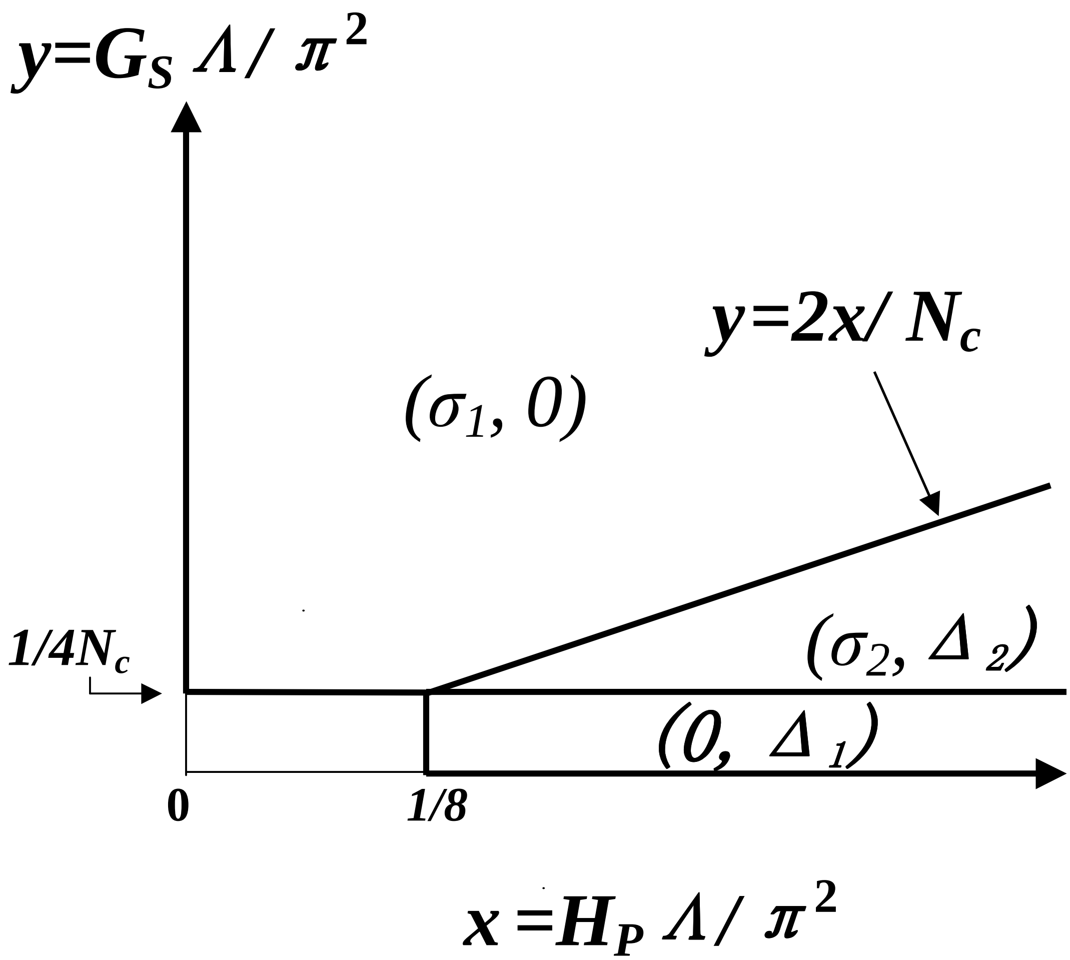


Fig. (d)

Main Conclusions

1. In all the models, pure $\langle\bar{q}q\rangle$ phase happens if $\frac{G_S}{H_S}$ (or $\frac{G_S}{H_P}$) $> \frac{2}{N_c}$ (also G_S must be large enough in 3D and 4D model).
2. The phases with condensates $\langle qq\rangle$, including pure $\langle qq\rangle$ phase and mixed phase with $\langle\bar{q}q\rangle$ and $\langle qq\rangle$, arise only if $\frac{G_S}{H_S}$ (or $\frac{G_S}{H_P}$) $< \frac{2}{N_c}$.
3. In 3D Gross-Neveu model, no mixed phase with $\langle\bar{q}q\rangle$ and $\langle qq\rangle$ exists for $N_c \leq 4$.

II. Motive and general approach

- In any four-fermion interaction model [1, 2], the couplings of $(qq)^2$-form and $(\bar{q}q)^2$-form can always coexist via the Fierz transformations, hence there must be interplay between the condensates $\langle\bar{q}q\rangle$ and $\langle qq\rangle$ in ground state of the system.
- In the vacuum, despite of absence of net fermions, based on a relativistic quantum field theory, it is possible that the condensates $\langle qq\rangle$ and $\langle\bar{q}\bar{q}\rangle$ are generated simultaneously.
- The mean field approximation has been taken. In this case, we have used the Fierz transformed four-fermion couplings in the Hartree approximation to avoid double counting [3].
- In selecting the couplings of $(qq)^2$-form, we always simulate $SU(N_c)$ gauge interaction, where two fermions are attractive in the antisymmetric $\frac{N_c(N_c-1)}{2}$ -plet.
- Euclidean momentum cutoffs in 3D and 4D models have been used so as to maintain Lorentz invariance of effective potentials in the vacuum.
- In massless fermion limit, all the discussions can be made analytically.
- The coupling constants G_S and H_S (or H_P) are viewed as independent parameters.

III. 4D Nambu-Jona-Lasinio model

With 2 flavors and N_c color massless fermions, the Lagrangian

$$\mathcal{L} = \bar{q}i\gamma^\mu\partial_\mu q + G_S[(\bar{q}q)^2 + (\bar{q}i\gamma_5\vec{\tau}q)^2] + H_S\sum_{\lambda_A}(\bar{q}i\gamma_5\tau_2\lambda_A q^C)(\bar{q}^C i\gamma_5\tau_2\lambda_A q), \qquad (1)$$

where the fermion fields q are in the doublet of $SU_f(2)$ and the N_c-plet of $SU_c(N_c)$, i.e.

$$q = \begin{pmatrix} u_i \\ d_i \end{pmatrix} \quad i = 1, \cdots, N_c, \qquad (2)$$

q^C is the charge conjugate of q and $\vec{\tau}=(\tau_1,\tau_2,\tau_3)$ are the Pauli matrices acting in two-flavor space. The matrices λ_A run over all the antisymmetric generators of $SU_c(N_c)$.
Assume that the four-fermion interactions can lead to the scalar condensates

$$\langle \bar{q}q\rangle = \phi \tag{3}$$

with all the N_c color fermion entering them, and the scalar color $\frac{N_c(N_c-1)}{2}$ -plet difermion and di-antifermion condensates (after a global $SU_c(N_c)$ transformation)

$$\langle \bar{q}^C i\gamma_5\tau_2\lambda_2 q\rangle = \delta, \quad \langle \bar{q}i\gamma_5\tau_2\lambda_2 q^C\rangle = \delta^*, \tag{4}$$

with only two color fermions enter them. The corresponding symmetry breaking is that $SU_{fL}(2)\otimes SU_{fR}(2)\to SU_f(2)$, $SU_c(N_c)\to SU_c(2)$, and a "rotated" electric charge $U_{\tilde{Q}}(1)$ and a "rotated" quark number $U'_q(1)$ leave unbroken. It should be indicated that in the case of vacuum, the Goldstone bosons induced by spontaneous breaking of $SU_c(N_c)$ could be some combinations of difermions and di-antifermions.
Define that

$$\sigma=-2G_S\phi, \ \ \Delta=-2H_S\delta, \ \ \Delta^*=-2H_S\delta^*. \tag{5}$$

With standard technique and a 4D Euclidean momentum cutoff Λ [4], we obtain the relativistic effective potential

$$\begin{aligned} V_4(\sigma,|\Delta|) \ &= \ \frac{\sigma^2}{4G_S}+\frac{|\Delta|^2}{4H_S}-\frac{1}{4\pi^2}\left[(N_c\sigma^2+2|\Delta|^2)\Lambda^2-(N_c-2)\frac{\sigma^4}{2}\left(\ln\frac{\Lambda^2}{\sigma^2}+\frac{1}{2}\right)\right. \\ &\left. -(\sigma^2+|\Delta|^2)^2\left(\ln\frac{\Lambda^2}{\sigma^2+|\Delta|^2}+\frac{1}{2}\right)\right]. \end{aligned} \tag{6}$$

The ground states of the system, i.e. the minimum points of $V_4(\sigma,|\Delta|)$, will be at

$$(\sigma,|\Delta|)=\begin{cases}(0, \quad \Delta_1) \\ (\sigma_2, \ \Delta_2) \\ (\sigma_1, \quad 0)\end{cases} \text{if} \begin{cases} \dfrac{H_S\Lambda^2}{\pi^2}>\dfrac{1}{2}, \qquad 0 \ \le \ \dfrac{G_S}{H_S}<\dfrac{1}{1+(N_c-2)\dfrac{H_S\Lambda^2}{\pi^2}} \\ \dfrac{1}{1+(N_c-2)\dfrac{H_S\Lambda^2}{\pi^2}} \ < \ \dfrac{G_S}{H_S}<\dfrac{2}{N_c} \\ \dfrac{G_S\Lambda^2}{\pi^2}>\dfrac{1}{N_c}, \qquad \dfrac{G_S}{H_S}>\dfrac{2}{N_c}\end{cases}, \tag{7}$$

Eq.(7) gives the phase diagram Fig.(a) of the 4D NJL model.

IV. 2D Gross-Neveu model

The Lagrangian is expressed by

$$\mathcal{L}=\bar{q}i\gamma^\mu\partial_\mu q+G_S[(\bar{q}q)^2+(\bar{q}i\gamma_5\vec{\tau}q)^2]+H_S(\bar{q}i\gamma_5\tau_S\lambda_A q^C)(\bar{q}^C i\gamma_5\tau_S\lambda_A q), \tag{8}$$

All the denotations are the same as ones in 4D NJL model, except that in 2D space-time

$$\gamma^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \ \ \gamma^1=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=-C, \ \ \gamma_5=\gamma^0\gamma^1$$

and $\tau_S = (\tau_0 \equiv 1, \tau_1, \tau_3)$ are flavor-triplet symmetric matrices. It is indicated that the product matrix $C\gamma_5\tau_S\lambda_A$ is antisymmetric.

Assume that the four-fermion interactions could lead to the scalar quark-antiquark condensates

$$\langle \bar{q}q \rangle = \phi, \tag{9}$$

which will break the discrete symmetries

$$\chi_D : q(t,x) \stackrel{\chi_D}{\to} \gamma_5 q(t,x),$$

$$\mathcal{P}_1 : q(t,x) \stackrel{\mathcal{P}_1}{\to} \gamma^1 q(t,-x),$$

and that the coupling with H_S can lead to the scalar color $\frac{N_c(N_c-1)}{2}$ -plet difermion condensates and the scalar color anti- $\frac{N_c(N_c-1)}{2}$ -plet di-antifermion condensates (after a global transformation in flavor and color space)

$$\langle \bar{q}^C i\gamma_5 1_f \lambda_2 q \rangle = \delta, \quad \langle \bar{q} i\gamma_5 1_f \lambda_2 q^C \rangle = \delta^* \tag{10}$$

which will break discrete symmetries Z_3^c (center of $SU_c(3)$) and Z_2^f (center of $SU_f(2)$), besides χ_D and $\mathcal{P}_1$. Noting that in a 2D model, no breaking of continuous symmetry needs to be considered on the basis of Mermin-Wagner-Coleman theorem [5].

The model is renormalizable. In the space-time dimension regularization approach, we can write down the renormalized $\mathcal{L}$ in $D = 2 - 2\varepsilon$ dimension space-time by the replacements

$$G_S \to G_S M^{2-D} Z_G, \; H_S \to H_S M^{2-D} Z_H,$$

with the scale parameter M, the renormalization constants Z_G and Z_H. In addition, the γ^μ in $\mathcal{L}$ will become $2^{D/2} \times 2^{D/2}$ matrices.

Define the order parameters

$$\sigma = -2G_S M^{2-D} Z_G \phi, \quad \Delta = -2H_S M^{2-D} Z_H \delta, \tag{11}$$

which will be finite if Z_G and Z_H are selected so as to cancel the UV divergences in ϕ and δ. In the minimal substraction scheme,

$$Z_G = 1 - \frac{2N_c G_S}{\pi}\frac{1}{\varepsilon}, \; Z_H = 1 - \frac{4H_S}{\pi}\frac{1}{\varepsilon}. \tag{12}$$

By similar derivation to the one made in Ref.[6], the corresponding renormalized effective potential in the mean field approximation up to one-loop order becomes

$$\begin{aligned} V_2(\sigma, |\Delta|) \; &= \; \frac{\sigma^2}{4G_S} - \frac{\sigma^2}{2\pi}\left(2\ln\frac{\bar{M}^2}{\sigma^2+|\Delta|^2} + (N_c-2)\ln\frac{\bar{M}^2}{\sigma^2} + N_c\right) \\ &+ \frac{|\Delta|^2}{4H_S} - \frac{|\Delta|^2}{\pi}\left(\ln\frac{\bar{M}^2}{\sigma^2+|\Delta|^2} + 1\right), \; \bar{M}^2 = 2\pi e^{-\gamma} M^2, \end{aligned} \tag{13}$$

where γ is the Euler constant. The ground states of the system i.e. the minimal points of $V_2(\sigma, |\Delta|)$ will be at

$$(\sigma, |\Delta|) = \begin{cases} (0, \quad \Delta_1) \\ (\sigma_2, \; \Delta_2) \\ (\sigma_1, \quad 0) \end{cases} \text{if} \; \begin{cases} G_S/H_S = 0 \\ 0 < G_S/H_S < 2/N_c \\ G_S/H_S > 2/N_c \end{cases} \tag{14}$$

Eq.(14) gives the phase diagram Fig.(b) of 2D GN model.
In 2D case, the G_S-H_S phase structure has the following feature:

1. *The pure $\langle qq\rangle$ phase $(0,\Delta_1)$ could appear only if $G_S/H_S=0$;*

2. *Formations of the condensates do not call for that the coupling constant G_S and H_S have some lower bounds.*

V. 3D Gross-Neveu model

The Lagrangian is expressed by

$$\mathcal{L}=\bar{q}i\gamma^\mu\partial_\mu q+G_S[(\bar{q}q)^2+(\bar{q}\vec{\tau}q)^2]+H_P\sum_{\lambda_A}(\bar{q}\tau_2\lambda_A q^C)(\bar{q}^C\tau_2\lambda_A q), \tag{15}$$

where $\gamma^\mu(\mu=0,1,2)$ are taken to be 2×2 matrices

$$\gamma^0=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\ \gamma^1=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix},\ \gamma^2=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}=C.$$

It is noted that the product matrix $C\tau_2\lambda_A$ is antisymmetric, and since without the "γ_5" matrix, the only possible color $\dfrac{N_c(N_c-1)}{2}$ -plet difermion interaction channel is pseudoscalar one. The condensates $\langle\bar{q}q\rangle$ will break

$$\text{time reversal symmetry}\ \ \mathcal{T}: q(t,\vec{x})\to\gamma^2 q(-t,\vec{x}),$$

$$\text{special parity}\ \ \mathcal{P}_1: q(t,x^1,x^2)\to\gamma^1 q(t,-x^1,x^2),$$

$$\text{special parity}\ \ \mathcal{P}_2: q(t,x^1,x^2)\to\gamma^2 q(t,x^1,-x^2).$$

The difermion condensates $\langle\bar{q}^C\tau_2\lambda_2 q\rangle$ (after a global rotation in the color space) will break

$$SU_c(N_c)\to SU_c(2)$$

and leave a "rotated" electrical charge $U_{\tilde{Q}}(1)$ and a "rotated" fermion number $U'_q(1)$ unbroken. It also breaks

$$\text{parity}\ \ \mathcal{P}: q(t,\vec{x})\to\gamma^0 q(t,-\vec{x})$$

and this shows pseudoscalar feature of the difermion condensates.

Define the order parameters in the 3D GN model

$$\sigma=-2G_S\langle\bar{q}q\rangle,\quad \Delta=-2H_P\langle\bar{q}^C\tau_2\lambda_2 q\rangle, \tag{16}$$

on bases of the same method used in Ref.[7], we find out the effective potential in the mean field approximation

$$\begin{aligned}V_3(\sigma,|\Delta|) &= \frac{\sigma^2}{4G_S}+\frac{|\Delta|^2}{4H_P}-\frac{1}{\pi^2}(N_c\sigma^2+2|\Delta|^2)\Lambda\\ &+\frac{1}{3\pi}\left[6\sigma^2|\Delta|+2|\Delta|^3+(N_c-2)\sigma^3+2\theta(\sigma-|\Delta|)(\sigma-|\Delta|)^3\right],\end{aligned} \tag{17}$$

where Λ is a 3D Euclidean momentum cutoff. The ground states of the system correspond to the least value points of $V_3(\sigma, |\Delta|)$ which will respectively be at

$$(\sigma, |\Delta|) = \begin{cases} (0, \Delta_1), & \text{if } \frac{G_S}{H_P} < \frac{2}{N_c}, \ \frac{H_P \Lambda}{\pi^2} > \frac{1}{8}, & \text{for } N_c \le 4 \\ \begin{cases} (0, \Delta_1), \\ (\sigma_2, \Delta_2), \end{cases} & \text{if } \frac{G_S}{H_P} < \frac{2}{N_c}, \ \frac{H_P \Lambda}{\pi^2} > \frac{1}{8}, \ \frac{G_S \Lambda}{\pi^2} \begin{cases} < \frac{1}{4N_c}, \\ > \frac{1}{4N_c}, \end{cases} & \text{for } N_c > 4 \\ (\sigma_1, 0), & \text{if } \frac{G_S}{H_P} > \frac{2}{N_c}, \qquad \frac{G_S \Lambda}{\pi^2} > \frac{1}{4N_c}, & \text{for all } N_c \end{cases} \tag{18}$$

Eq.(18) gives the $G_S - H_P$ phase diagrams Fig.(c) and Fig.(d) of the 3D GN model.

VI. Summary

- Present research deepens our theoretical understanding of the four-fermion interaction models:
  1. Even in vacuum, it is possible that the difermion condensates are generated as long as the coupling constants of the difermion channel are strong enough (bigger than zero or some finite values).
  2. Interplay between the condensates $\langle \bar{q}q \rangle$ and $\langle qq \rangle$ mainly depends on G_S/H_S (or G_S/H_P), the ratio of the coupling constants of scalar fermion-antifermion channel and scalar (or pseudoscalar) difermion channel.
  3. In all the discussed 2-flavor models, if G_S/H_S $(G_S/H_P) > 2/N_c$, the ratio of the color numbers of the fermions entering into the condensates $\langle qq \rangle$ and $\langle \bar{q}q \rangle$, (and also with sufficiently large G_S in 4D and 3D model), then only pure $\langle \bar{q}q \rangle$ condensates phase may exist. Below $2/N_c$, (and also with sufficiently large H_S or H_P in 4D or 3D model), one will always first have a mixed phase with condensates $\langle \bar{q}q \rangle$ and $\langle qq \rangle$, then a pure $\langle qq \rangle$ condensate phase, except that in the 3D GN model, no the mixed phase appears when $N_c \le 4$.
- In view of absence of $\langle qq \rangle$ condensates in vacuum of QCD, the result here also implies a real restriction to any given two-flavor QCD-analogue NJL model: in such model, the derived smallest ratio G_S/H_S via the Fierz transformation in the Hartree approximation must be bigger than 2/3 [4].

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